Revista Brasiliense de Engenharia e Física Aplicada

Stellar Interiors from Luminothermic Modeling:

A Pedagogic Approach

Nilo Serpa, Ph.D., research engineer in cosmology and quantum gravity
GAUGE-F Scientific Researches, Brazil; Faculdades ICESP.

Received: 20 Nov 2016 __ / Accepted: 02 Apr 2017 __ / Published: 16 Jun 2017 __.

Abstract: The teaching of astronomy in Brazil has primacy for the quality and objectivity, both in the undergraduate and graduate schools, especially in the classrooms of the School of Astronomy at the Valongo Observatory, belonging to the Federal University of Rio de Janeiro. However, and this is a generalized fact in the process of teaching/learning at all levels, there is not much incentive in Brazil to develop the capacity of one freely mine a problem and propose an original solution. The requirements are very much about what is already known, without giving space and time to the fundamental questions underlying the current astrophysical and cosmological models. As a consequence, very little real contribution comes out from our offices and instruments. This pedagogical article is an attempt to show how to propose an original approach to teaching and to exercise freedom of thought in the search of the main elements for the construction of knowledge about the basics on stellar interiors, an issue far more theoretical than one can imagine at first. Also it summarizes aspects and results arising from simulations on polytropic models. The simulations emerged from what I called "luminothermic fitting". The study was developed on the comparison between the adaptive stellar interior model given by Novotny (1973) and the toy model constructed from a FORTRAN program, introducing initially the same parameters. I considered the abundances of hydrogen (X) and metals (Z) worth 0.70 and 0.02 respectively.

Key words: polytropic star, stellar interior, opacity, thermodynamic equilibrium, luminothermic fitting.

.

Nomenclature

G: gravitational constant

P: pressure

$M(r)$: global mass as a function of the stellar radius

Greek letters

$\rho\ (r)$: density as a function of the stellar radius

σ: Stefan-Boltzmann constant

1. Introduction

Doing well to characterize present model as a "toy" model, I remember that, although modern stellar evolution models contain much more physics details, the

Corresponding author: Nilo Sylvio Costa Serpa, Ph.D., Professor, research fields: quantum gravity, quantum computing, cosmology and thermal systems engineering. E-mail: nilo.serpa@icesp.edu.br.

empirical limits remain the same: nobody went inside a star to measure its physical properties; stars are tremendously turbulent bodies and unimaginably hot. Apart the emerging field of asteroseismology [9], enabling us to perform a more direct observational study of stellar interiors, the inner side of the stars are effectively unseen to external observers, so that all the information we receive from them originates in their atmospheres. Everything we really have is a consistent set of hypotheses and presuppositions based in great part on very simplistic propositions; such propositions were applied and treated in the classical literature ([3], [4], [11], [13]). Of course, there are milestone works on stellar structure elsewhere that deal with the subject in great depth ([1], [2], [5], [6], [7], [8], [9], [10], [12]). Thus, what I shall explain is a simple and heuristic mathematical model to undergraduate students in order to aim them to construct the first knowledge on stellar astrophysics. I strongly recommend to the students the textbook introduction on the basic elements of fundamental astronomy and astrophysics by Bohm-Vitense (1992) [3], mainly the first half of the book 1, explaining how stellar radii, luminosities, masses and temperatures are measured or deduced.

The study of stellar interiors has the aim to determine the internal variation of the main physical properties of the stars. To simplify this difficult task, we introduce some approximate representations to quantify the physics within these wonderful objects. Combining the general polytropic modeling with few additional assumptions, we can obtain beautiful and consistent results.

The concept of polytropic star refers to very simplified models about the internal variability of the main properties of the stars as we shall see ahead. Such models are represented by systems of equations that express the relevant physical processes related to those properties. The simplified character of these models allows one to obtain analytical or numerical solutions for the representative equations, solutions which describe the variations in question.

In stellar interiors it is supposed that 1) - conditions change slowly over many photon mean free paths and 2) - stellar matter is close to local thermodynamic equilibrium. These assumptions lead to a step-by-step smooth transition of state, becoming simple the integration of the transfer equation. Despite the fact that this is a job with a high degree of uncertainty, the determination of temperature is crucial in stellar physics, both to locate the star in the HR diagram and to support studies on abundances and gravity. So, once we consider the mass and chemical composition fixed, the challenge I took was to obtain from a certain FORTRAN program (named Interior2007, implemented at Valongo Observatory, 2007) a realistic model of stellar interior by the adjustment of the effective temperature, preserving the dimensions of the object and the surface density by the concomitant adjustment of luminosity. This kind of tuning was called luminothermic fit. This option seemed quite reasonable when you consider that in the chosen example, the error in the stellar radius remained on average below 5%. The distancing of the initial temperature was within two orders of magnitude, being quite understated. Also the decay of the radial luminosity remained over a gentle relaxation.

The idea of including changes in the program Interior2007 in order to make realistic models of stellar interiors by luminothermic adjustment, with no change in radius, mass or surface density, is perfectly feasible for simulations with low error in the radius and, at least in principle, for stars with masses not much larger than the Sun, that is, stars having surface temperatures of a few thousand degrees Kelvin. Although I have no rights over the distribution of the FORTRAN program used, I do not see difficult to build a similar code in any other language for the calculations discussed here, easily found in the literature mentioned at the end of this article.

It is noteworthy that the luminothermic fit does not intend in any way to exhaust the subject, just start the complex task of building feasible models.

2. Methodology

The prior existence of a temperature gradient in the star is the guaranty that the flow of energy in one direction overcomes the energy flow in the opposite direction, generating a resulting non-zero stream-oriented from the hottest regions to the cooler. In addition, among the three different mechanisms by which this flow of energy can be realized — radiation, conduction and convection —, radiation is by far the most important.

2.1 - The essential on polytropic approach

An extensive discussion on polytropic models was made in my doctoral thesis [14], from which I reproduce here a fragment. To understand the degree of simplification of the theory in focus, we should start from the premise of the hydrostatic equilibrium, according to which the force of gravity and the pressure across the layershell with radius *r* obey the stationary condition

$$\begin{cases} \dfrac{dM(r)}{dr} = 4\pi r^2 \rho(r) \\ \dfrac{dP}{dr} = -\dfrac{GM(r)\rho(r)}{r^2}, \end{cases} \tag{1}$$

from which we deduce, for a spherical star,

$$\frac{d}{dr}\left(\frac{dP}{dr}\frac{r^2}{\rho}\right) = -4\pi G r^2 \rho. \tag{2}$$

In present explanation, P is the pressure, $\rho(r)$ is the density at radius r, $M(r)$ is the global mass at radius r, and G is the usual gravitational constant. From the second equation of the system (1),

$$\frac{r^2}{G\rho(r)}\frac{dP}{dr} = -M(r);$$

$$-\frac{dM(r)}{dr} = \frac{r^2}{G\rho}\frac{d^2P}{dr^2} + \frac{2r}{G\rho}\frac{dP}{dr} - \frac{r^2}{G\rho^2}\frac{d\rho}{dr}\frac{dP}{dr};$$

$$\frac{r^2}{G\rho^2}\frac{d\rho}{dr}\frac{dP}{dr} - \frac{r^2}{G\rho}\frac{d^2P}{dr^2} - \frac{2r}{G\rho}\frac{dP}{dr} = 4\pi r^2 \rho(r);$$

$$\frac{r^2}{\rho^2}\frac{d\rho}{dr}\frac{dP}{dr} - \frac{r^2}{\rho}\frac{d^2P}{dr^2} - \frac{2r}{\rho}\frac{dP}{dr} = 4\pi G r^2 \rho.$$

But,

$$\frac{d}{dr}\left(\frac{dP}{dr}\frac{r^2}{\rho}\right) = \frac{d^2P}{dr^2}\frac{r^2}{\rho} + \frac{2r}{\rho}\frac{dP}{dr} - \frac{dP}{dr}\frac{r^2}{\rho^2}\frac{d\rho}{dr};$$

$$\frac{d}{dr}\left(\frac{dP}{dr}\frac{r^2}{\rho}\right) = -4\pi G r^2 \rho. \tag{3}$$

Following, since the equation of state that describes the polytropic star has the form $P = K\rho^{1+1/n}$, with n (the polytropic index) and K being constants throughout the star,

$$\frac{dP}{dr} = \frac{n+1}{n}K\rho^{1/n}\frac{d\rho}{dr};$$

$$\frac{d}{dr}\left(\frac{n+1}{n}K\rho^{1/n}\frac{d\rho}{dr}\frac{r^2}{\rho}\right) = -4\pi G r^2 \rho;$$

$$\frac{d}{dr}\left(r^2\rho^{-1}\rho^{1/n}\frac{d\rho}{dr}\right) = -\frac{n}{n+1}\frac{4\pi G r^2 \rho}{K}. \tag{4}$$

To establish the boundary conditions of the problem, it is interesting to adopt the following replacements:

$$r = ax,\ \rho = by^n.$$

To the center of the star, $r \to 0$, $x \to 0$. In these circumstances, it is convenient that $y \to 1$, so that

$$\rho = \rho_c y^n \rightarrow \rho = \rho_c \text{ (at the center).}$$

Similarly, for the surface we have

$$r \rightarrow R, x \rightarrow x(R) = R/a,$$

$$\text{with } \rho \rightarrow 0 \ (y \rightarrow 0).$$

These new variables must be substituted in equation

$$\frac{d}{dr}\left(r^2\rho^{-1}\rho^{1/n}\frac{d\rho}{dr}\right) = -\frac{n}{n+1}\frac{4\pi G r^2 \rho}{K}, \text{ in such}$$

manner that

$$dr = adx,$$

$$\frac{1}{a} = \frac{dx}{dr},$$

$$\frac{1}{adx} = \frac{1}{dr},$$

and

$$\frac{d\rho}{dr} = \frac{d\rho}{dy}\frac{dy}{dr} = n\rho_c y^{n-1}\frac{dy}{dr} = n\rho_c y^{n-1}\frac{dy}{adx},$$

we have

$$\frac{d}{adx}\left(a^2x^2\rho^{\frac{1}{n}-1}n\rho_c y^{n-1}\frac{1}{a}\frac{dy}{dx}\right) = -\frac{n}{n+1}\frac{4\pi G r^2\rho}{K};$$

$$\frac{d}{adx}\left(\cancel{a}^2x^2\left(\rho_c y^n\right)^{\frac{1}{n}-1}\cancel{n}\cancel{\rho_c}y^{n-1}\frac{1}{a}\frac{dy}{dx}\right) =$$

$$= -\frac{4\pi G}{K}\frac{1}{n+1}\cancel{n}\cancel{a}^2x^2\cancel{\rho_c}y^n;$$

$$\frac{d}{a^2dx}\left(x^2\rho_c^{\frac{1-n}{n}}\underbrace{y^{1-n}y^{n-1}}_{1}\frac{dy}{dx}\right) = -\frac{4\pi G}{K}\frac{1}{n+1}x^2y^n;$$

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -\frac{4\pi G}{K}\rho_c^{\frac{n-1}{n}}\frac{y^n}{n+1}a^2.$$

For formal convenience, it is necessary that

$$a^2 = \frac{(n+1)K}{4\pi G\rho_c^{\frac{n-1}{n}}},$$

which leads to $\frac{4\pi G}{K}\rho_c^{\frac{n-1}{n}}\frac{a^2}{n+1} = 1$. Therefore,

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -\frac{4\pi G}{K}\rho_c^{\frac{n-1}{n}}\frac{y^n}{n+1}a^2,$$

reduces to

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^n;$$

$$\frac{1}{x^2}2x\frac{dy}{dx} + \frac{1}{x^2}x^2\frac{d^2y}{dx^2} + y^n = 0;$$

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^n = 0. \qquad (5)$$

The solution of this last equation, called the Lane-Emden equation, determines the internal structure of the polytropic stars. Polytropic modeling of stars is closely related to predictions provided by Eddington's assumption that gas pressure and radiation (pushing everything outward the star), and gravity (pulling everything toward the center of the star) establish an internal stellar balance. Eddington's theory works with pressure, temperature and density, linked by the law of perfect gases. However, it is important to note that Lane-Emden equation is deduced without any considerations about the transport of energy in question because it assumes that this transport is implicitly determined by the polytropic equation itself.

2-2. The simulations

In general, all simulations performed were done by the same way. The initial step was to match the table of Novotny with the tables generated by the FORTRAN program, using as key-field the radius of the first table, obtaining by approximation the best correspondence between the registers and keeping the average error in radius < 5%. The first generations of tables showed quantities without physical meaning, such as negative luminosities. Assuming the polytropic approach and the application of the luminothermic fitting, that is, the tuning of luminosity and effective temperature, the iterations have taken place until the complete eradication of inappropriate quantities. Thereafter, my work focused on the establishment of the combined values of L and T_{eff}, so that I got as better luminothermic approach to the Novotny modeling the hereunder scenario:

1) $M = 1M_{Sun}$;

2) $L = 0.38L_{Sun}$;

3) $R = 1.0763R_{Sun}$;

4) $T_{eff} = 3380°K$;

5) $\rho_s = 0.002146$ (at the surface);

6) $X = 0.70$;

7) $Z = 0.02$.

The luminosity L of a star, that is, the rate of energy emitted per unit time at all frequencies and directions crossing a spherical surface, is related to the bolometric magnitude $\tilde{M}$ in terms of solar luminosity by the expression

$$\frac{L}{L_{Sun}} = 10^{-0.4(\tilde{M}-\tilde{M}_{Sun})}, \qquad (6)$$

where $L_{Sun} = 3.83\times10^{33}\, erg/s$.

The effective temperature is given by

$$T_{eff}^4 = \frac{\pi}{\sigma}\int B_\nu(T)d\nu,$$

where ν is the frequency and $B_\nu(T)$ is the Planck function according to

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1},$$

with h being the Planck constant, c the velocity of light and k = 1.38x10-16 erg/K. The Stefan-Boltzmann constant is defined as

$$\sigma = 5.67\times10^{-5}\, erg\ cm^{-2}\mathrm{K}^{-4}s^{-1}.$$

The above scenario refers to a reddish star, colder, therefore, than the sun, but with the same mass occupying a volume a bit higher. Due to this small volume difference, it did not seem unreasonably the searching for a viable model without changing the weight-spatial dimensions of the Novotny model.

The interior modeling was extremely sensitive to luminothermic variations. A small change in effective temperature or luminosity was sufficient to cause physical inconsistencies. Thus, it was necessary to integrate the magnitudes involved repeatedly for fixed values of luminosity and effective temperature very close to each other, until the rescue of consistent scenarios. Additional comparative tests with the same basic results, however, not published here, were made for a similar luminothermic model, assuming L = 0.578 L_{Sun} and R = 1.021 R_{Sun} (Schwarzschild, 1965).

3. Results

Figures 1, 2 and 3 show qualitatively reasonable values for density, temperature and pressure by comparing the toy model with the Novotny model, highlighting the moderate discrepancy in the values close to the center. Indeed, the betterment of the central discrepancy occurred with the reduction in surface luminosity concomitant to the reduction in effective temperature, although it is risky to say without more accurate investigations this continues to happen for cooler stars maintaining physical sense. A fact is that for a star somewhat weaker, with L=0.28L_{Sun} and T_{eff} = 2800°K, this trend of improvement in modeling the central region of stars seems to be confirmed in accordance with the obtained graphics (Figures 11 and 12).

Figure 4 shows a comparison between the temperature curves in both models, but with the inclusion of a logarithmic regression from the toy model. Note that the regression curve gives a reasonable approximation to the Novotny model mainly from the middle of the radius till the surface of the star, serving also as a possible theoretical basis for future implementations in FORTRAN program with respect to the refinement of the modeling of semi-solar stars. The program was modified to produce one more column in the final file containing the logarithmic adjustment to be applied to radial temperature from the toy model (Figure 5).

Figure 6 displays the luminosity curve of the Novotny model, showing semi-logistic shape. Notice that although a range of values is not properly adjusted by the FORTRAN program, the luminosity in toy model shows a similar way (Figure 7) and fits very well to a sixth order polynomial regression. Unlike the logarithmic regression, which was applied as a good approximation to many luminothermic variations in a range of effective temperatures from $2600°K$ to $6600°K$, no exhaustive tests were done in this study on the polynomial regression of luminosity in other semi-solar stars.

Figure 8 fits gas pressure and radioactive pressure in the toy model, while Figure 9 shows that the electron scattering dominates as an opacity[1] factor only in a very short fraction of the radius (0.1%), and was immediately supplanted by the bound-free transition and then by free-free transition at 8% of the radius.

Figure 10 shows the rates of energy generation over the temperature with a sensitive record of the viability of the CNO cycle from approximately 2×10^{7}°K, corresponding to a fraction of the radius of about 10% from the center. As might be expected, the proton-proton chain seems the more important rule valid for stars of mass less than or equal to the solar mass.

4. Conclusion

Despite the fact of the great complexity of stellar interiors, due to the high degree of interdependence of the variables in question, this article showed the possibility of simplifications in the initial study of stars with semi-solar dimensions, analyzing a hypothetical structure clearly plausible. Although qualitatively reasonable results have been found, I cannot guarantee that the program FORTRAN be able to properly emulate all the data of Novotny because of some functional differences found in the simulations produced. Better luminothermic approximations can be consummated with superior computational resources to allow parallel simulations to high processing speeds, even capable of performing real-time covariant analysis of the impact caused by small simultaneous changes in the values of each intervening variable. I suggest to those interested in astrophysics the experience with similar initiatives, though their academic environments may not be very enthusiastic for creative and independent research.

5. References

[1] Benz, A. 2002. *Plasma Astrophysics: Kinetic Processes in Solar and Stellar Coronae*. Kluwer Academic Publishers, New York.

[1] Opacity, or transparence, is an important property of the stellar medium, determining the efficiency of the energy transport by photons from the inner shells of the star. A good approach on opacity is given in reference [12].

[2] Boer K. S. de, Seggewiss, W. 2008. *Stars and Stellar Evolution*. EDP Sciences, France.

[3] Bohm-Vitense, E. 1992. *Introduction to Stellar Astrophysics*. Cambridge University Press, vols. 1, 2 and 3.

[4] Chandrasekhar, S. 1938. *An Introduction to the Study of Stellar Structure*. The University of Chicago Press, Illinois, USA.

[5] Collins, II, G. 2003. *The Fundamentals of Stellar Astrophysics*. Case Western Reserve University, Ohio, USA.

[6] Guzik, J. 2010. "Recent Advances in Modeling Stellar Interiors", arXiv:1005.5406.

[7] Hansen, C., Kawaler, S., Trimble, V. 2004. *Stellar Interiors: Physical Principles, Structure and Evolution*. Springer-Verlag, New York, USA.

[8] Kippenhahn, R., Weigert, A. 1994. *Stellar Structure and Evolution*. Springer-Verlag, Berlin.

[9] Leblanc, F. *An Introduction to Stellar Astrophysics*. John Wiley & Sons, Sussex, UK.

[10] Maciel, W. 2016. *Introduction to Stellar Structure*. Springer International Publishing, Switzerland.

[11] Novotny, E. 1973. *Introduction to Stellar Atmospheres and Interiors*. Oxford University Press, 2nd edition.

[12] Rogers, F., Iglesias, C. 1994. "Astrophysical Opacity", *Science*, New Series, vol. 263, nº 5143, 50-55.

[13] Schwarzschild, M. 1965. *Structure and Evolution of the Stars*. Dover, 2nd edition, New York.

[14] Serpa, N., *Sur l'Entropie Contrôlée des Systèmes* [..]. Ph.D. Thesis, L'Université Libre des Sciences de L'Homme de Paris, 127p (2014).

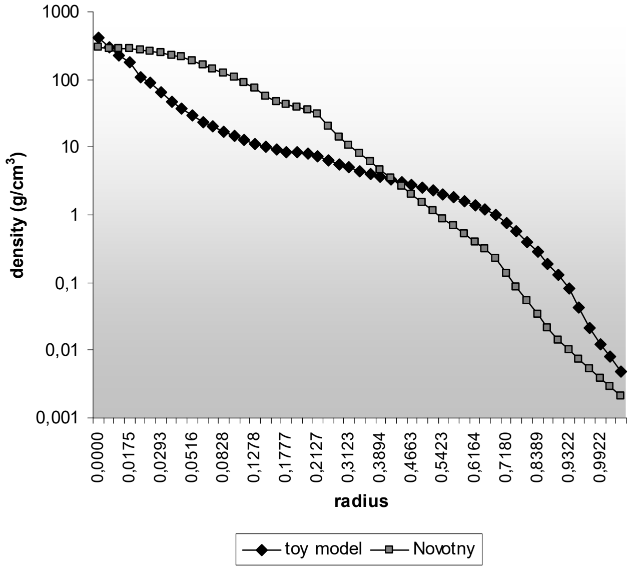

Figure 2 - Density in Novotny and in the toy-model.

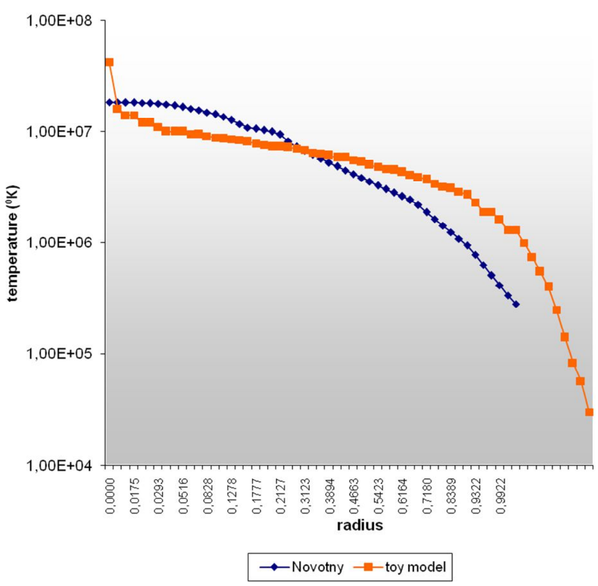

Figure 1- Temperature in Novotny and in the toy-model.

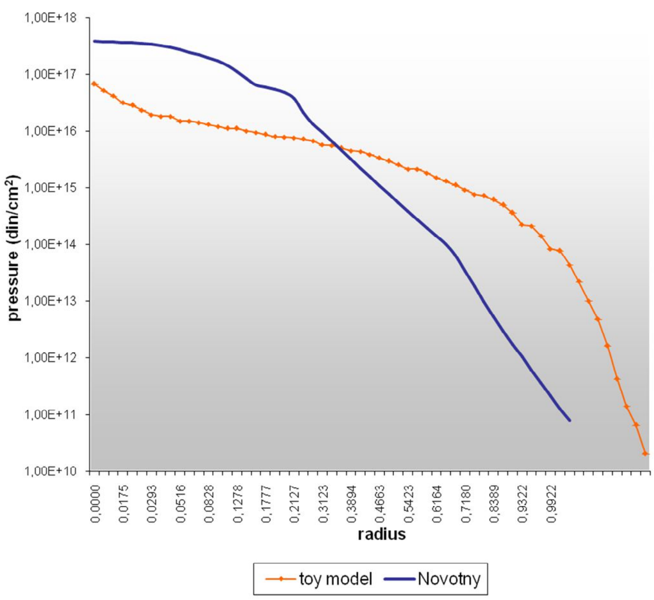

Figure 3 - Pressure in Novotny and in the toy-model.

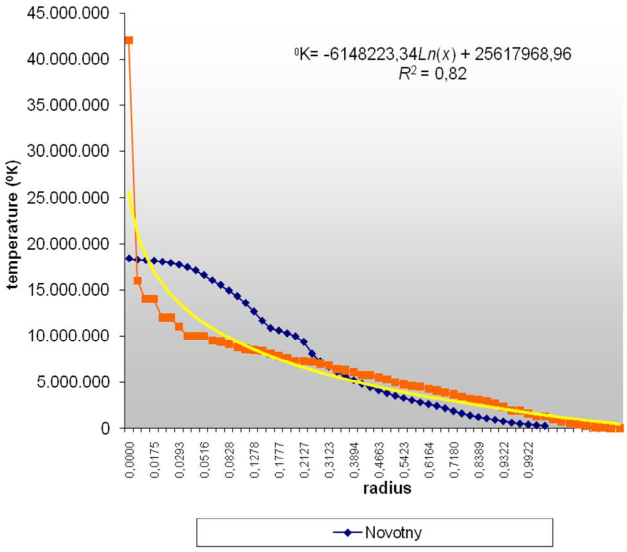

Figure 4 -Temperature in Novotny and in the toy-model with logarithmic regression from the latter.

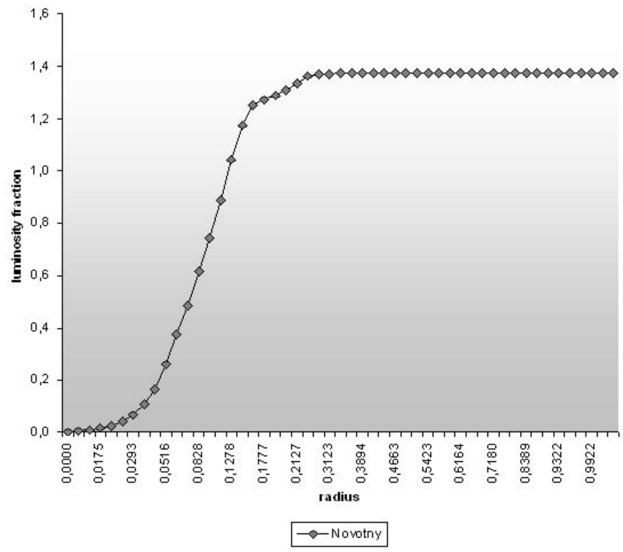

Figure 6 - Luminosity in Novotny.

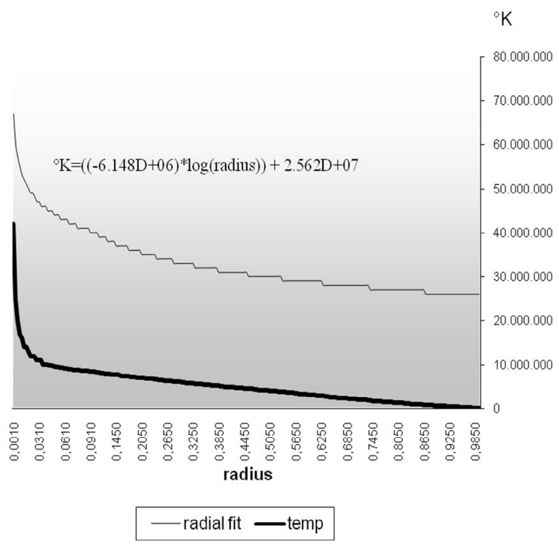

Figure 5 - Logarithmic radial fit for temperature accordingly the toy-model.

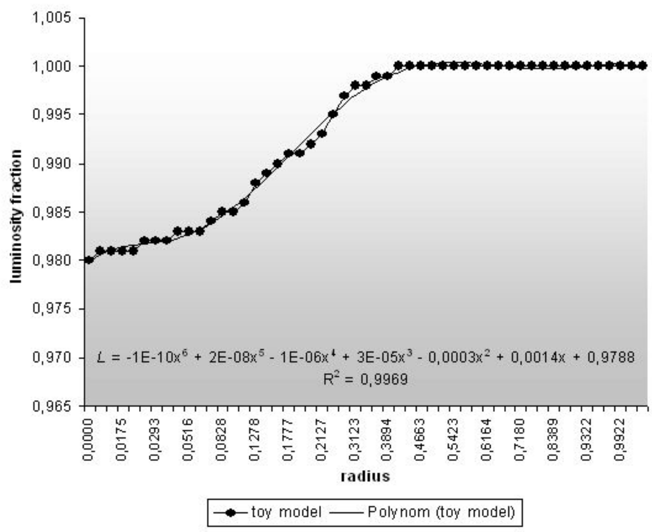

Figure 7 - Luminosity in the toy model with polynomial regression of 6^{th} order.

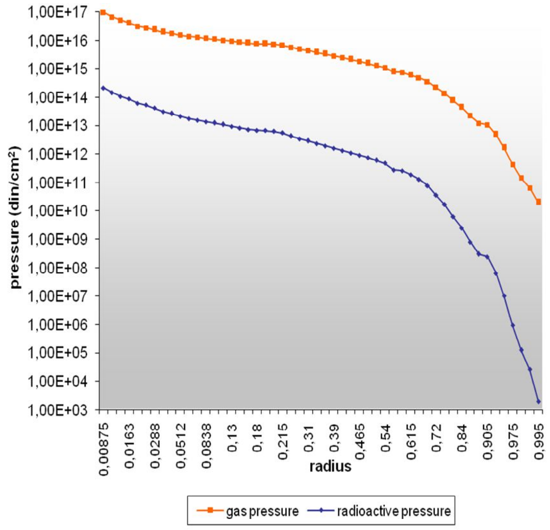

Figure 8 - Gas pressure and radioactive pressure in the toy model.

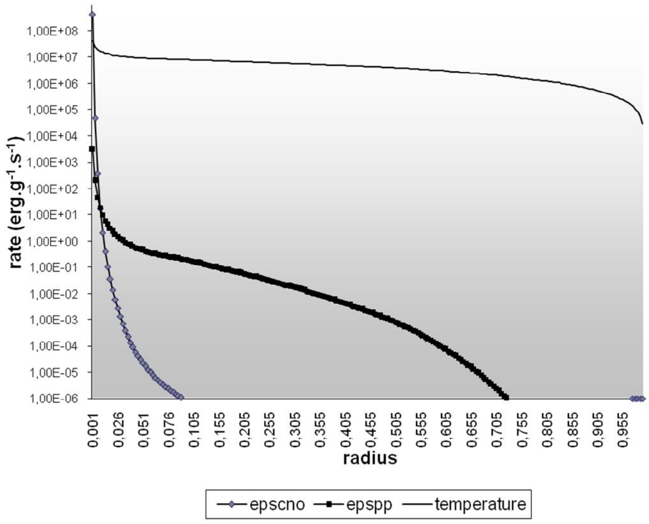

Figure 10 - Energy production in the toy model.

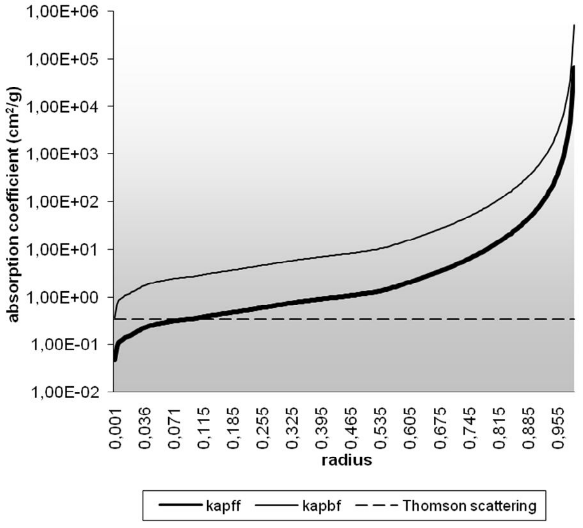

Figure 9 - Opacity in the toy model.

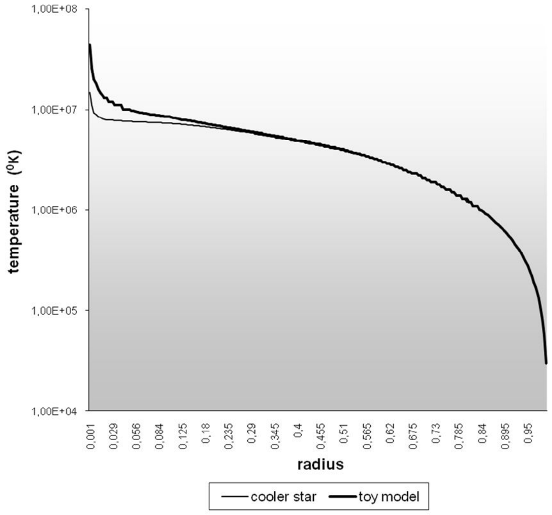

Figure 11 - Temperature in the model with L=0.28L_{Sun} and T_{eff} = 2800ºK, and in the toy model.

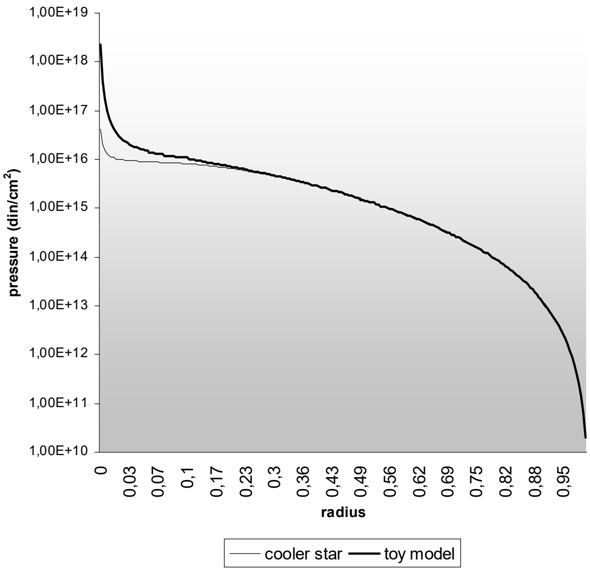

Figure 12 - Pressure in the model with L=0.28L_{Sun} and T_{eff} = 2800^oK , and in the toy model.